How many physical qubits are needed *exactly* for fault-tolerant quantum computing?

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December 28, 2021

Abstract

Physical implementation of quantum computing is a continuing challenge. The main obstacle is understanding how to map abstract qubits and quantum operations into physical hardware without decoherence, that is, losing quantum information. A solution is proposed here via the theory of Nash embedding.

1 Introduction

To manufacture quantum computers, abstract *logical* qubits require realization as *physical* qubits in the classical world. But this mapping has to be done in a way that the resulting physical qubits are *fault-tolerant* [1, 2], that is, they stay coherent when a quantum computation is performed on them.

The idea that one can perform a quantum computation on physical qubits in the classical world requires elaboration. *Logical quantum computing* occurs in the quantum physical realm, a notion described mathematically by the pair $\left\{\mathbb{C}P^{n-1}, Q\right\}$ where the complex projective Hilbert space $\mathbb{C}P^{n-1}$ is the quantum register of n qubits, the state of which is transformed by the unitary operation Q. To implement the logical quantum computation $\left\{\mathbb{C}P^{n-1}, Q\right\}$ in physical hardware in the classical world requires mapping into *physical quantum computation*, a notion captured mathematically by the pair $\left\{\mathbb{R}^d, R\right\}$ where the Euclidean space $\mathbb{R}^d$ is the physical register for the n qubits, the state of which is transformed by an operation R.

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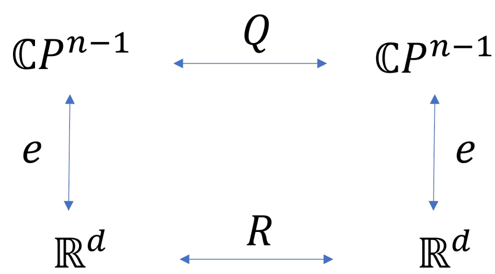

Figure 1: Nash embedding e of the initial and final state of the quantum register, together with unitarity of quantum computations Q, requires that the physical quantum computation R is in fact an orthogonal, reversible computation

2 Fault-tolerant physical quantum computing

For the physical quantum computation to be fault-tolerant, the mapping from $\{\mathbb{C}P^{n-1}, Q\}$ to $\{\mathbb{R}^d, R\}$ should be robust against fundamental notions of faults that arise when traversing the classical-quantum physical divide. From a general mathematical perspective, faults in the traversal of the classical-quantum physical divide are due to changes in the topology, geometry, or differential structure. If the traversal function from $\{\mathbb{C}P^{n-1}, Q\}$ to $\{\mathbb{R}^d, R\}$ preserves topology, geometry, and differential structure, than the resulting physical quantum computation will be considered fault-tolerant.

The class of functions $e : \{\mathbb{C}P^{n-1}, Q\} \longrightarrow \{\mathbb{R}^d, R\}$ preserving topology, geometry, and differential structure are Nash embeddings [3]. A Nash embedding is a one-to-one map that is a homeomorphism (preserves topological features), diffeomorphism (preserves differential structures), and an isometry (preserves distances). These properties of e imply, as shown in Figure 1, that the physical quantum computation R emulating the logical quantum computation Q is necessarily a reversible computation, that is, an orthogonal transformation.

A Nash embedding gives the exact number of fault-tolerant physical qubits needed to implement coherent quantum computation in hardware in the form of the number d. As per Nash's theorem

$$d = \max\left\{\frac{k(k+5)}{2}, \frac{k(k+3)}{2} + 5\right\}. \tag{1}$$

where k is the dimension of the quantum register $\mathbb{C}P^{n-1}$ *as a Riemannian manifold* [4]. This number is $k = 2^{n+1} - 2$ (where n is the number of qubits). This means that one qubit register $\mathbb{C}P^1$ maps to the fault-tolerant physical register $\mathbb{R}^{10}$; therefore, 1 logical qubit maps into 10 fault-tolerant physical qubits. Similarly, 2 logical qubits map into 19 fault-tolerant physical qubits, 3 logical qubits map into 52 fault-tolerant physical qubits, and 4 logical qubits map into 168 fault-tolerant physical qubits. This number grows big fast. Twenty logical qubits give $k = 2,097,150$, and hence map into $2,199,024,304,125$ fault-tolerant physical qubits!

3 Conclusion

Regardless of this dramatic increase in the size of the fault-tolerant physical qubit register, any meaningful effort in its design and manufacturing should account for these *Nash values*. There are bottom-up, heuristic efforts underway for fault-tolerant physical qubit development commercially at Dwave Systems, Honeywell, and IonQ, to name a few. Efforts in the loading classical data into qubits [5] for machine learning can also benefit greatly from the two-way robustness of Nash embedding in traversing the classical-quantum divide, not to mention the possibility of suppressing decoherence caused by quantum measurement. The top-down , systematic theory of Nash embedding will serve as a benchmark in the search for materials that can be used to produce fault-tolerant physical qubits, hastening the day of the fault-tolerant quantum computer.

References

[1] Shor, P., *Fault-tolerant quantum computation*, Proceedings of the 37th Symposium on Foundations of Computing, IEEE Computer Society Press, pp. 56-65 (1996).

[2] Egan, L., Debroy, D.M., Noel, C. et al. *Fault-tolerant control of an error-corrected qubit*, Nature 598, 281–286 (2021).

[3] Nash, J. *The imbedding problem for Riemannian manifolds*, Annals of Mathematics, 63 (1): 20–63 (1956).

[4] Bengtsson, I., Życzkowski, K., *Geometry of Quantum States: An Introduction to quantum entanglement*, Publisher: Cambridge University Press (2007).

[5] Lloyd, S. et al., *Quantum embeddings for machine learning*, preprint available at arXiv:2001.03622 [quant-ph]